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Carleman Estimates with a Second Large Parameter

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Carleman estimates are an indispensable tool for proving uniqueness of continuation for solutions to partial differential equations with non-analytic coefficients. We prove a new Carleman estimate with two large parameters for operators with time independent coefficients which combines features of estimates given by D. Tataru (1995, *Comm. Partial Differential Equations* **20,** 855–884) and V. Isakov (1998, On the uniqueness of continuation for a thermoelasticity system, preprint).

1. INTRODUCTION AND MAIN RESULT

Uniqueness of continuation results for solutions to linear partial differential equations is of importance in many branches of applied mathematics, in particular in control theory and inverse problems. Holmgren's theorem is the fundamental result which gives uniqueness of continuation across noncharacteristic surfaces for scalar equations or systems of equations with analytic coefficients. For scalar equations with non-analytic coefficients the problem is more difficult. The classical result is Hörmander's theorem [H1, Theorem 8.9.1], which establishes uniqueness of continuation across a level surface of a C^2 function ψ which satisfies a certain convexity condition with respect to the operator in an open connected set Ω. The proof is based on Carleman estimates, i.e., an estimate of the form

$$\tau^{2m-2|\alpha|-1}\int |D^\alpha(e^{\tau\varphi}u)|^2 \le c\int |e^{\tau\varphi}P(x,D)u|^2, \qquad \tau \ge \tau_0 \quad (1.1)$$

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for $u \in C_0^\infty(\Omega)$ where $\varphi = e^{\lambda\psi}$ is a strongly pseudo-convex function for $\lambda \geq \lambda_0$, $P(x, D)$ is an operator of order m, and $|\alpha| \leq m - 1$.

Hörmander's theorem cannot be applied to most relevant systems of equations because of multiple characteristics. However, if the principal part of the system is scalar, then it can lead to uniqueness results for systems as well; see, e.g., [E-I-N-T].

Often, the practical circumstances yield operators with time-independent coefficients. For second order operators with time-independent real valued coefficients Tataru's result [T1] gives the conclusion of Holmgren's theorem. The proof of his result is based on Carleman estimates with respect to the space variable and regularization by means of a Gaussian regularizer with respect to time. These estimates look like

$$\tau^{2m-2|\alpha|-1} \int |D^\alpha e^{-(1/2\tau)D_t^2}(e^{\tau\varphi}u)|^2 \leq c\Bigg(\int |e^{-(1/2\tau)D_t^2} e^{\tau\varphi} P(x, D)u|^2 + e^{-d\tau} \int |e^{\tau\varphi} P(x, D)u|^2 + e^{-d\tau} \sum_{|\alpha| \leq m-1} \tau^{2m-2-2|\alpha|} |D^\alpha(e^{\tau\varphi}u)|^2 \Bigg). \quad (1.2)$$

Again, for principally scalar systems an application of Tataru's result will give uniqueness results [E-I-N-T].

In case of the thermo-elastic system

$$\begin{aligned} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \operatorname{div}(\alpha(x)\nabla\theta) &= 0 \\ \theta_t - \Delta\theta - \operatorname{div}(\alpha(x)\nabla w_t) &= 0 \end{aligned} \quad (1.3)$$

which is not principally scalar V. Isakov managed to prove a uniqueness result by deriving a Carleman estimate with a second large parameter [I]. He uses the fact that the strongly pseudo-convex function is given by $\varphi = e^{\lambda\psi}$ to obtain an estimate like

$$\lambda \int (\tau\lambda e^{\lambda\psi})^{2m-2|\alpha|-1} |D^\alpha(e^{\tau\varphi}u)|^2 \leq c \int |e^{\tau\varphi} P(x, D)u|^2, \qquad \lambda \geq \lambda_0,\ \tau \geq \tau_0(\lambda) \quad (1.4)$$

for the Laplacian and the heat operator. This estimate appears to be much more complicated than (1.1); however, when we ignore λ and consider it to be a constant we get (1.1) back. The interesting feature of this estimate is the additional λ up front.

In this paper we will combine the approaches by Tataru and Isakov and obtain a Carleman estimate carrying a second large parameter and a Gaussian regularizer with respect to time. This estimate will be the foundation of the following local uniqueness result for the thermo-elastic system (1.3). For a precise formulation and its proof we refer to [E-L-T, Theorem 11.3].

THEOREM 1.0. *Assume that $\alpha \in C^2$ is time-independent and that S is a timelike C^2-surface with respect to the wave operator $\gamma\partial_t^2 - \Delta$ at x_0. Moreover, assume that $(w, \theta) \in H^3$ is a solution to the system* (1.3) *which vanishes on one side of S. Then $w \equiv 0$ and $\theta \equiv 0$ in a neighborhood of x_0.*

This local result improves the theorem by Isakov [I] and gives rise to a better global uniqueness theorem as well, see [E-L-T, Theorem 4.2.2]. Actually, questions about unique continuation for this thermo-elastic system led originally to this research.

In order to formulate our result we introduce some notation. Let $x = (t, x') \in \mathbf{R}^{n+1}$. The corresponding Fourier variables are denoted by $\xi = (\xi_0, \xi')$. Let $I \subset \mathbf{R}$ be an open, connected set in time and $\Omega \subset \mathbf{R}^n$ be an open, connected set in space. Moreover, let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

(with $D_j = -i\partial_j = -i\partial/\partial x_j$) be a linear partial differential operator of order m defined on some neighborhood of $I \times \Omega$. We denote its principal symbol by $p(x, \xi)$, i.e.,

$$p(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

In addition, let ψ be an function defined in a neighborhood of $I \times \Omega$ and set

$$\varphi = e^{\lambda\psi} - 1.$$

For the composition of the differential operator $P(x, D)$ and the multiplication by the weight function $e^{\tau\varphi}$ we have the formula

$$e^{\tau\varphi} P(x, D) u = P(x, D + i\tau\nabla\varphi(x))(e^{\tau\varphi} u) \tag{1.5}$$

which suggests that the operator

$$P(x, D + i\tau\nabla\varphi(x)) = \sum_{|\alpha| \leq m} a_\alpha(x)(D + i\tau\nabla\varphi(x))^\alpha$$

is of further interest. In this context we introduce the symbol

$$\begin{aligned} p_\varphi(x, \xi, \tau) &= p(x, \xi + i\tau\nabla\varphi(x)) = p(x, \xi + i\kappa(x)\nabla\psi(x)) \\ &= p_\psi(x, \xi, \kappa), \end{aligned}$$

where we set $\kappa(x) = \lambda\tau e^{\lambda\psi(x)}$.

By H^s we denote the L_2 based Sobolev space of order s. For $u \in H^s$ we introduce the weighted norm

$$|u|_{s,\tau}^2 = \frac{1}{(2\pi)^{n+1}} \int \left(\tau^2 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi,$$

where $\hat{u}$ denotes the Fourier transform of u. For positive integer s an equivalent norm is given by

$$|u|_{s,\tau}^2 = \sum_{|\alpha| \leq s} \tau^{2s-2|\alpha|} |D^\alpha u|_0^2.$$

For a positive integer k there are two more weighted norms

$$|u|_{k,\kappa}^2 = \sum_{|\alpha| \leq k} \int \kappa(x)^{2k-2|\alpha|} |D^\alpha u|^2$$

and

$$|u|_{k,\kappa^*}^2 = \sum_{|\alpha| \leq k} \int \kappa(x)^{2k-1-2|\alpha|} |D^\alpha u|^2$$

for $u \in H^k$. Finally, for $s, k \in \mathbf{R}$ we will also need an anisotropic norm

$$|u|_{s,k,\tau}^2 = \frac{1}{(2\pi)^{n+1}} \int \left(\tau^2 + \xi_0^2\right)^s \left(\tau^2 + |\xi|^2\right)^k |\hat{u}(\xi)|^2 \, d\xi.$$

Now we state our main result.

THEOREM 1.1. *Let $P(x, D)$ be a differential operator on $I \times \Omega$ of order m with time independent coefficients that are C^∞ in the principal part and bounded otherwise. Assume that $p(x, \xi) \neq 0$ for $\xi_0 = 0$ and $\xi' \neq 0$ and $x' \in \Omega$. Moreover, let $\psi(x)$ be an analytic function on a neighborhood of $I \times \Omega$ such that $\nabla\psi(x) \neq 0$ and $p_\psi(x, \xi, \eta)$ considered as a polynomial in η has no multiple real root for $\xi_0 = 0$ and $\xi' \neq 0$ and $x \in \overline{I \times \Omega}$.*

Then there exist a constant $c > 0$ and a positive function $d(\lambda) > 0$ such that

$$\sqrt{\lambda}\, |e^{-(1/2\tau)D_t^2} e^{\tau\varphi} u|_{m,\kappa^*} \leq c\Big(|e^{-(1/2\tau)D_t^2} e^{\tau\varphi} P(x, D) u|_0 + e^{-d\tau} |e^{\tau\varphi} P(x, D) u|_0 + e^{-d\tau} |e^{\tau\varphi} u|_{m-1,\tau}\Big)$$

for $u \in H^{m-1}$ compactly supported in $I \times \Omega$ provided $\lambda \geq \lambda_0$ and $\tau \geq \tau_0(\lambda)$ and the right hand side is finite.

Here we make some comments. Comparing this estimate with (1.4) we notice that the powers on the left hand side are exactly the same. The norm $|\cdot|_{m,\kappa^*}$ is just used to capture those properties. Moreover, on the other hand our estimate is very similar to Tataru's Carleman estimate (1.2). We can express the same estimate using different norms.

COROLLARY 1.2. *Let all assumptions of Theorem* 1.1 *be satisfied. Then there exist a constant* $c > 0$ *and a positive function* $d(\lambda) > 0$ *such that*

$$\sqrt{\lambda}|e^{-(1/2\tau)D_t^2}e^{\tau\varphi}u|_{m,\kappa} \leq c\Big(|\kappa^{1/2}e^{-(1/2\tau)D_t^2}e^{\tau\varphi}P(x,D)u|_0 + e^{-d\tau}|e^{\tau\varphi}P(x,D)u|_0 + e^{-d\tau}|e^{\tau\varphi}u|_{m-1,\tau}\Big)$$

for $u \in H^{m-1}$ *compactly supported in* $I \times \Omega$ *provided* $\lambda \geq \lambda_0$ *and* $\tau \geq \tau_0(\lambda)$ *and the right hand side is finite.*

However, our new estimate needs a stronger assumption than (1.1) and (1.2). Instead of the strong pseudo-convexity condition we require that the polynomial $p_\psi(x,\xi,\cdot)$ does not have any real double roots. This assumption is reminiscent of the assumptions of Calderon's uniqueness theorem [H2, Theorem 28.1.1]. Also, we assume that the operator is elliptic in space. Moreover, the assumptions of Theorem 1.1 are satisfied for a large class of second order operators.

Remark 1.1. Assume that $P(x,D)$ is a second order operator such that the principal symbol has real C^∞ coefficients and is elliptic with respect to ξ' on a neighborhood of $I \times \Omega$. Then all the assumptions of Theorem 1.1 are satisfied as long as ψ has analytic non-characteristic level surfaces in a neighborhood of $I \times \Omega$.

This follows from the fact that p_ψ is a second degree polynomial in $i\eta$ with real coefficients and lead coefficient $p(x,\nabla\psi(x))$.

We like to present the main ideas of the proof of Theorem 1.1. At first we will derive an inequality for the symbol $p_\varphi(x,\xi,\tau)$. We will show that there exists a constant $C > 0$ such that

$$C\big(|\xi|^2 + \kappa^2\big)^m \leq |p_\varphi(x,\xi,\tau)|^2 + \xi_0^{2m} + \frac{\kappa}{2i\lambda}\Big\{\overline{p_\varphi(x,\xi,\tau)}, p_\varphi(x,\xi,\tau)\Big\}$$

for $\lambda \geq \lambda_0$ and $\tau \geq 0$ where $\{\cdot,\cdot\}$ denotes the Poisson bracket.

In the next step we will apply the Fefferman–Phong inequality and get a preliminary Carleman estimate

$$C\sqrt{\lambda}|v|_{m,\kappa^*} \leq |P(x, D + i\tau\nabla\varphi(x))v|_0 + \sqrt{\lambda}|D_t^m v|_{0,\kappa^*},$$

where v is a C^∞-function compactly supported in a neighborhood of $I \times \Omega$ provided $\lambda \geq \lambda_0$ and $\tau \geq \tau_0(\lambda)$.

In the last step we will substitute $v = e^{-D_t^2/2\tau} e^{\tau\varphi} u$ into this estimate and obtain the main result.

Section 2 is dedicated to the proof of Theorem 1.1. It is divided into three subsections; each corresponds to the step outlined above. Section 3 contains the proofs of technical results used in Section 2. Finally, Section 4 states some results for operators where the time variable is not distinguished.

2. PROOF OF THEOREM 1.1

In the following we denote the r neighborhood of I in $\mathbf{R}$ by I_r, i.e.,

$$I_r = \{t \in \mathbf{R} : \operatorname{dist}(t, I) < r\}$$

and we denote the r neighborhood of Ω in $\mathbf{R}^n$ by Ω_r, i.e.,

$$\Omega_r = \{x \in \mathbf{R}^n : \operatorname{dist}(x, \Omega) < r\}.$$

2.1. *An Inequality for the Symbol $p_\varphi(x, \xi, \tau)$*

PROPOSITION 2.1. *Let $p(x, \xi)$ be a the principal symbol of a differential operator of order m on $I \times \Omega$ such that $p(x, \xi) \neq 0$ for $\xi_0 = 0$ and $\xi' \neq 0$ and $x \in I_{4r} \times \Omega_r$.*

Moreover, assume $\psi(x) \in C^2(\overline{I_{4r} \times \Omega_r})$ is chosen such that $\nabla\psi(x) \neq 0$ and $p_\psi(x, \xi, \eta)$ considered as a polynomial in η has no multiple real roots for $\xi_0 = 0$, $\xi' \neq 0$, and $x \in I_{4r} \times \Omega_r$.

Then there exists a constant $K > 0$ such that

$$8K^2\big(|\xi|^2 + \kappa(x)^2\big)^m \leq |p_\varphi(x, \xi, \tau)|^2 + \xi_0^{2m} + \frac{\kappa(x)}{2i\lambda}\Big\{\overline{p_\varphi(x, \xi, \tau)}, p_\varphi(x, \xi, \tau)\Big\} \quad (2.1)$$

for $\lambda \geq \lambda_0$ and $x \in I_{4r} \times \Omega_r$. The function $\kappa(x)$ is the one introduced in Section 1, i.e., $\kappa(x) = \tau\lambda e^{\lambda\psi(x)}$.

Proof. Let S^n be the unit sphere in $\mathbf{R}^n \times \mathbf{R}$, i.e.,

$$S^n = \Big\{(\xi, \eta) : |\xi|^2 + \eta^2 = 1,\ \xi_0 = 0\Big\}.$$

Then there exists a constant $C_1 > 0$ such that

$$|p_\psi(x,\xi,\eta)|^2 + \eta^2 \left| \frac{\partial}{\partial \eta} p_\psi(x,\xi,\eta) \right|^2 > C_1$$

for $(\xi, \eta) \in S^n$ and $x \in I_{4r} \times \Omega_\tau$. For η bounded away from zero this follows from the assumption that the polynomial $p_\psi(x, \xi, \eta)$ has no multiple zeros. For $\eta = 0$ it follows from the spatial ellipticity condition and by continuity it is valid in a small neighborhood of $\eta = 0$.

Denoting the unit sphere in $\mathbf{R}^{n+1} \times \mathbf{R}$ by S^{n+1} we proceed with

$$|p_\psi(x,\xi,\eta)|^2 + \eta^2 \left| \frac{\partial}{\partial \eta} p_\psi(x,\xi,\eta) \right|^2 + \xi_0^2 > C_2$$

for $(\xi, \eta) \in S^{n+1}$, $x \in I_{4r} \times \Omega_r$, and some new constant $C_2 > 0$. By homogeneity we conclude

$$|p_\psi(x,\xi,\eta)|^2 + \eta^2 \left| \frac{\partial}{\partial \eta} p_\psi(x,\xi,\eta) \right|^2 + \xi_0^{2m} \geq C_2\big(|\xi|^2 + \eta^2\big)^m \tag{2.2}$$

for all (ξ, η) and $x \in I_{4r} \times \Omega_r$. Next we compute the Poisson bracket (see [H2, Sect. 28.2])

$$\begin{aligned} \frac{1}{2i\eta}\Big\{\overline{p_\psi(x,\xi,\eta)}, p_\psi(x,\xi,\eta)\Big\} = &\sum_{j,k=0}^{n} \partial_{jk}\psi(x)\overline{p_\psi^{(j)}(x,\xi,\eta)}p_\psi^{(k)}(x,\xi,\eta) \\ &+ \frac{1}{\eta}\operatorname{Im}\sum_{j=0}^{n} \overline{p_{(j),\psi}(x,\xi,\eta)}p_\psi^{(j)}(x,\xi,\eta) \end{aligned} \tag{2.3}$$

and by continuity and homogeneity there exists a constant $C_3 > 0$ such that

$$\frac{\eta}{2i\lambda}\Big\{\overline{p_\psi(x,\xi,\eta)}, p_\psi(x,\xi,\eta)\Big\} \geq -\frac{C_3}{\lambda}\big(|\xi|^2 + \eta^2\big)^m \tag{2.4}$$

for all (ξ, η) and $x \in I_{4r} \times \Omega_r$.

Adding the two estimates (2.2) and (2.4) and choosing $\lambda \geq \lambda_0 = 2C_3/C_2$ we obtain

$$|p_\psi(x,\xi,\eta)|^2 + \eta^2 \left| \sum_{j=0}^{n} \partial_j \psi(x) p_\psi^{(j)}(x,\xi,\eta) \right|^2$$

$$+ \frac{\eta}{2i\lambda} \left\{ \overline{p_\psi(x,\xi,\eta)}, p_\psi(x,\xi,\eta) \right\} + \xi_0^{2m} \geq 8K^2 \left(|\xi|^2 + \eta^2 \right)^m$$

for all (ξ, η) and $x \in I_{4r} \times \Omega_r$ when we choose $K^2 = C_2/16$ and observe that

$$\frac{\partial}{\partial \eta} p_\psi(x,\xi,\eta) = i \sum_{j=0}^{n} \partial d_j \psi(x) p_\psi^{(j)}(x,\xi,\eta).$$

Now we set $\eta = \kappa(x)$ and observe that

$$\frac{\kappa(x)}{2i\lambda} \left\{ \overline{p_\varphi(x,\xi,\tau)}, p_\varphi(x,\xi,\tau) \right\} = \frac{\kappa(x)}{2i\lambda} \left\{ \overline{p_\psi(x,\xi,\kappa)}, p_\psi(x,\xi,\kappa) \right\}$$

$$+ \kappa(x)^2 \left| \sum_{j=0}^{n} \partial_j \psi(x) p_\psi^{(j)}(x,\xi,\kappa) \right|^2.$$

This last identity can be verified by an explicit computation, see [H3, Lemma 4.2]. ∎

2.2. *The Preliminary Carleman Estimate*

Here we will apply the Fefferman–Phong inequality to the inequality (2.1). However, at first we need to introduce a special class of symbols which is modeled along the lines of the properties of $p_\varphi(x,\xi,\tau)$. Throughout this subsection we assume that $\psi \in C_0^\infty(\mathbf{R}^{n+1})$.

DEFINITION 2.2. For $k \in \mathbf{R}$ we say $a(x,\xi) \in S^k_{\lambda,\tau}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ if $a(x,\xi) \in C^\infty(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ and

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C(\alpha,\beta) \lambda^{|\beta|} \left(\kappa(x) + |\xi| \right)^{k-|\alpha|},$$

$$x \in \mathbf{R}^{n+1},\ \xi \in \mathbf{R}^{n+1}, \quad (2.5)$$

where $\kappa(x)$ is the function introduced in Section 1, i.e., $\kappa = \tau\lambda e^{\lambda\psi}$.

This symbol space is a Frechet space and we like to point out that the seminorms $C(\alpha,\beta)$ do not depend on τ and λ. The operators with symbols in $S^k_{\lambda,\tau}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ are continuous operators between certain

Sobolev spaces, in particular we have for $a(x, \xi) \in S^k_{\lambda,\tau}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ the estimate

$$|a(x, D)u|_0 \le C|u|_{k, \kappa_{max}}, \tag{2.6}$$

where κ_{max} is the global maximum value of $\kappa(x)$.

In the following we show that this symbol class can be represented as $S(m, g)$ where g denotes a Riemannian metric on $T^*(\mathbf{R}^{n+1})$ and m is a weight function. This allows us to apply the results of the pseudo-differential calculus for symbols in $S(m, g)$ developed by Hörmander [H2, sects. 18.4–18.6], in particular the Weyl calculus.

We introduce a metric on $T^*(\mathbf{R}^{n+1})$ by

$$g = \lambda^2|dx|^2 + \frac{|d\xi|^2}{\kappa(x)^2 + |\xi|^2}.$$

This metric is slowly varying and the function $m(x, \xi) = (\kappa(x)^2 + |\xi|^2)^{k/2}$ is g continuous for $k \in \mathbf{R}$. With the notation used in [H2, Definition 18.4.2] the class of symbols defined above becomes

$$S^k_{\lambda,\tau}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}) = S\left(\left(\kappa(x)^2 + |\xi|^2\right)^{k/2}, g\right).$$

The dual quadratic form of g with respect to the symplectic form is

$$g^\sigma = \left(\kappa(x)^2 + |\xi|^2\right)|dx|^2 + \frac{|d\xi|^2}{\lambda^2}.$$

Moreover, one can show that the metric g is σ temperate and the function m is g, σ temperate for $\tau \ge \tau_1(\lambda)$. Furthermore, let

$$h(x, \xi)^2 = \sup_{(y,\eta) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}} \frac{g_{x,\xi}(y, \eta)}{g^\sigma_{x,\xi}(y, \eta)} = \frac{\lambda^2}{\kappa(x)^2 + |\xi|^2} \le 1.$$

The Fefferman–Phong inequality [H2, Theorem 18.6.8] specialized to this case is given in the following lemma.

LEMMA 2.3. *Assume that*

$$0 \le a(x, \xi) \in S\left(\frac{\kappa(x)^2 + |\xi|^2}{\lambda^2}, g\right)$$

for $\tau \geq \tau_1(\lambda)$. *Then*

$$(a^w(x, D)u, u) \geq -C|u|_0^2,$$

where $a^w(x, D)$ *is the operator corresponding to the Weyl symbol* $a(x, \xi)$ *and* $(\cdot, \cdot)$ *is the scalar product in* $L_2(\mathbf{R}^{n+1})$.

With this in mind we can prove the preliminary Carleman estimate.

PROPOSITION 2.4. *Let* $p(x, \xi)$ *be the principal symbol with coefficients* $a_\alpha \in C^\infty(I_{4r} \times \Omega_r)$ *of a linear partial differential operator* $P(x, D)$ *of order* m *such that* $p(x, \xi) \neq 0$ *for* $\xi_0 = 0$ *and* $\xi' \neq 0$ *and* $x \in I_{4r} \times \Omega_r$.

Moreover, assume $\psi(x) \in C_0^\infty(\mathbf{R}^{n+1})$ *is chosen such that* $\nabla\psi(x) \neq 0$ *for* $x \in I_{4r} \times \Omega_r$ *and* $p_\psi(x, \xi, \eta)$ *considered as a polynomial in* η *has no multiple real roots for* $\xi_0 = 0$ *and* $\xi' \neq 0$ *and* $x \in I_{4r} \times \Omega_r$.

Then, for $\lambda \geq \lambda_0$ *and* $\tau \geq \tau_3(\lambda)$ *we have*

$$K\sqrt{\lambda}|v|_{m,\kappa^*} \leq |P(x, D + i\tau\nabla\varphi(x))v|_0 + \sqrt{\lambda}|D_t^m v|_{0,\kappa^*} \tag{2.7}$$

for all $v \in C_0^\infty(I_{3r} \times \Omega)$. *Here* K *is the same constant as in Proposition* 2.1.

Proof. The coefficients of the principal part can be extended to functions $a_\alpha(x) \in C_0^\infty(\mathbf{R}^{n+1})$. Then

$$p_\varphi(x, \xi, \tau) \in S\Big(\big(\kappa(x)^2 + |\xi|^2\big)^{m/2}, g\Big).$$

Let $\rho \in \mathrm{C}_0^\infty(\mathrm{I}_{4\mathrm{r}} \times \Omega_\mathrm{r})$ such that $\rho \equiv 1$ on $\mathrm{I}_{3\mathrm{r}} \times \Omega$.

Then we deduce from formula (2.1) that

$$|p_\varphi(x, \xi, \tau)|^2 + \frac{1}{2i}\left\{\overline{p_\psi(x, \xi, \tau)}, p_\varphi(x, \xi, \tau)\right\} + \frac{\lambda}{\kappa}\xi_0^{2m} - 8K^2\frac{\lambda}{\kappa}\big(|\xi|^2 + \kappa^2\big)^m \geq 0$$

for $\lambda \geq \lambda_0$, $x \in I_{4r} \times \Omega_r$, and $\xi \in \mathbf{R}^{n+1}$. This inequality can be represented as

$$|p_\varphi(x, \xi, \tau)|^2 + \frac{1}{2i}\left\{\overline{p_\varphi(x, \xi, \tau)}, p_\varphi(x, \xi, \tau)\right\} + |q(x, \xi)|^2 - 8K^2|r(x, \xi)|^2 \geq 0. \tag{2.8}$$

Here

$$q(x,\xi) = \sqrt{\frac{\lambda}{\kappa(x)}}\,\xi_0^m \in S\Big(\big(\kappa(x)^2 + |\xi|^2\big)^{m/2}, g\Big)$$

and

$$r(x,\xi) = \sqrt{\frac{\lambda}{\kappa(x)}}\sum_{|\alpha|\le m}\kappa(x)^{m-|\alpha|}\xi^\alpha \in S\Big(\big(\kappa(x)^2 + |\xi|^2\big)^{m/2}, g\Big).$$

In order to obtain (2.8) we use also

$$|r(x,\xi)|^2 \le \frac{\lambda}{\kappa}\big(\kappa^2 + |\xi|^2\big)^m.$$

Let $p_\varphi^w(x,D,\tau)$ be the operator with the Weyl symbol $p_\varphi(x,\xi,\tau)$. According to the composition theorem of the Weyl calculus [H2, Theorem 18.5.4] the expression

$$|p_\varphi(x,\xi,\tau)|^2 + \frac{1}{2i}\big\{\overline{p_\varphi(x,\xi,\tau)}, p_\varphi(x,\xi,\tau)\big\}$$

is the symbol of $p_\varphi^w(x,D,\tau)^* p_\varphi^w(x,D,\tau)$ with an error in $S(\lambda^2(\kappa(x)^2 + |\xi|^2)^{m-1}, g)$. Furthermore since the symbols q and r are real valued, the symbols of $q^w(x,D)^* q^w(x,D)$ and $r^w(x,D)^* r^w(x,D)$ are $|q(x,\xi)|^2$ and $|r(x,\xi)|^2$, respectively, with an error in $S(\lambda^2(\kappa(x)^2 + |\xi|^2)^{m-1}, g)$. Consequently, the operator

$$\begin{aligned} s^w(x,D) &= p_\varphi^w(x,D,\tau)^* p_\varphi^w(x,D,\tau) + q^w(x,D)^* q^w(x,D) \\ &\quad - 8K^2 r^w(x,D)^* r^w(x,D) \end{aligned}$$

has the Weyl symbol

$$\begin{aligned} &|p_\varphi(x,\xi,\tau)|^2 + \frac{1}{2i}\big\{\bar p_\varphi(x,\xi,\tau), p_\varphi(x,\xi,\tau)\big\} + |q(x,\xi)|^2 \\ &\quad - 8K^2|r(x,\xi)|^2 \end{aligned}$$

$\in S((\kappa(x)^2 + |\xi|^2)^m, g)$ with an error in $S(\lambda^2(\kappa(x)^2 + |\xi|^2)^{m-1}, g)$.

Next we construct an operator to which we can apply the Fefferman–Phong inequality. The operator

$$t^w(x,D) = \big(|D|^2 + \kappa_{max}^2\big)^{(1-m)/2} s^w(x,D)\big(|D|^2 + \kappa_{max}^2\big)^{(1-m)/2}$$

has the Weyl symbol

$$t(x,\xi) = \left(|\xi|^2 + \kappa_{max}^2\right)^{(1-m)/2} s(x,\xi)\left(|\xi|^2 + \kappa_{max}^2\right)^{(1-m)/2}$$
$$\in S\left(\kappa(x)^2 + |\xi|^2, g\right)$$

with an error in $S(\lambda^2, g)$. This follows again from [H2, Theorem 18.5.4] since the operator $(|D|^2 + \kappa_{max}^2)^{(1-m)/2}$ has constant coefficients and its (Weyl) symbol is in $S((\kappa(x)^2 + |\xi|^2)^{(1-m)/2}, g)$. This last fact follows from the second statement in [H2, Lemma 18.4.3]. Applying Proposition 2.3 with $a(x,\xi) = \rho(x)t(x,\xi)\rho(x)/\lambda^2$ we have

$$(t^w(x,D)u,u) \geq -C_1\lambda^2|u|_0^2,$$

for some $C_1 > 0$ and $u \in C_0^\infty(I_{3r} \times \Omega)$ provided $\tau \geq \tau_1(\lambda)$. Here we use that $a^w(x,D)$ and $t^w(x,D)/\lambda^2$ have the same symbol modulo terms in $S(1,g)$ which follows from [H2, Theorem 18.5.4]. We like to point out that this constant can be larger than the constant in the Fefferman–Phong inequality since the "lower order terms" of $t^w(x,D)$, i.e., the terms with the Weyl symbol in $S(\lambda^2, g)$, are estimated by L^2 continuity.

Letting $u = (|D|^2 + \kappa_{max}^2)^{(m-1)/2}v$ in the estimate above we arrive at

$$|p_\varphi^w(x,D,\tau)v|_0^2 + |q^w(x,D)v|_0^2 \geq 8K^2|r^w(x,D)v|_0^2 - C_1\lambda^2|v|_{m-1,\kappa_{max}}^2. \tag{2.9}$$

Applying [H2, Theorem 18.5.10] and formula (2.6) adjusted to symbols in $S(\lambda(\kappa(x)^2 + |\xi|^2)^{(m-1)/2}, g)$ we have

$$|p_\varphi^w(x,D,\tau)v|_0^2 \leq 2|P(x,D + i\tau\nabla\varphi(x))v|_0^2 + C_2\lambda^2|v|_{m-1,\kappa_{max}}^2$$

and

$$|q^w(x,D)v|_0^2 \leq 2|q(x,D)v|_0^2 + C_3\lambda^2|v|_{m-1,\kappa_{max}}^2$$

and on the other hand

$$4K^2|r(x,D)v|_0^2 \leq 8K^2|r^w(x,D)v|_0^2 + C_4\lambda^2|v|_{m-1,\kappa_{max}}^2.$$

These two estimates are owed to the fact that the Weyl operator and the standard pseudo differential operator of the same symbol have the same "principal part."

Hence, formula (2.9) implies

$$|P(x,D + i\tau\varphi(x))v|_0^2 + |q(x,D)v|_0^2 \geq 2K^2\lambda|v|_{m,\kappa^*}^2 - C_5\lambda^2|v|_{m-1,\kappa_{max}}^2$$

when we observe that $|r(x, D)v| = \sqrt{\lambda}|v|_{m,\kappa^*}$ and $C_5 = (C_1 + C_2 + C_3 + C_4)/2$.

We will show that for sufficiently large τ one-half of the first term in the right hand side absorbs the other term in the right hand side. This can be seen as follows. Using the definitions of the norms we obtain

$$\begin{aligned}
&2K^2\kappa(x)^{2m-1-2|\alpha|} - C_5\lambda\kappa_{max}^{2m-2-2|\alpha|} \\
&\quad = 2K^2\kappa(x)^{2m-1-2|\alpha|}\left(1 - \frac{C_5\lambda\kappa_{max}^{2m-2-2|\alpha|}}{2K^2\kappa(x)^{2m-1-2|\alpha|}}\right) \\
&\quad = 2K^2\kappa(x)^{2m-1-2|\alpha|}\left(1 - \frac{C_5 e^{\lambda(2m-2-2|\alpha|)\psi_{max}}}{2K^2\tau e^{\lambda(2m-1-2|\alpha|)\psi(x)}}\right) \\
&\quad \geq K^2\kappa(x)^{2m-1-2|\alpha|}
\end{aligned}$$

provided

$$\tau \geq \tau_2(\lambda) = \max_{|\alpha|\leq m-1} \frac{C_5 e^{\lambda(2m-2-2|\alpha|)\psi_{max}}}{K^2 e^{\lambda(2m-1-2|\alpha|)\psi_{min}}}.$$

Here ψ_{max} and ψ_{min} denote the global maximum and minimum value of ψ, respectively. Finally, we choose $\tau_3(\lambda) = \max\{\tau_1(\lambda), \tau_2(\lambda)\}$. ∎

The following corollary presents the same estimate using a different weighted norm.

COROLLARY 2.5. *Under the same assumptions as in Proposition* 2.4 *we have the estimate*

$$K\sqrt{\lambda}|v|_{m,\kappa} \leq |\kappa^{1/2}P(x, D + i\tau\nabla\varphi(x))v|_0 + \sqrt{\lambda}|D_t^m v|_0 \qquad (2.10)$$

for $v \in C_0^\infty(I_{3r} \times \Omega)$ *provided* $\lambda \geq \lambda_0$ *and* $\tau \geq \tau_3(\lambda)$.

Proof. We use estimate (2.7) for the function $\kappa^{1/2}v$ instead of v. Of course, the constant K in (2.10) is different from the constant used in (2.7). ∎

The next corollary is an estimate in terms of the anisotropic norm introduced in the Introduction. It will be used in the following subsection.

COROLLARY 2.6. *Let all the assumptions of Proposition* 2.4 *be satisfied. Then for all positive integers N there exists a constant* $c(\lambda)$ *such that*

$$|u|_{-N,m,\tau} \leq c(\lambda)\left(\sqrt{\tau}|P(x, D + i\tau\nabla\varphi(x))u|_{-N,0,\tau} + |u|_{-N+m,m-1,\tau}\right)$$

for $u \in C^\infty(I_{3r} \times \Omega)$ *provided* $\lambda \geq \lambda_0$ *and* $\tau \geq \tau_3(\lambda)$.

Proof. We will use estimate (2.7) with $v = (D_t^2 + \tau^2)^{-N/2}u$. At first we observe that

$$c(\lambda)|u|_{-N,m,\tau} \leq \sqrt{\tau\lambda}\,|v|_{m,\kappa^*}.$$

On the right hand side we observe that

$$\begin{aligned}&|P(x, D + i\tau\nabla\varphi(x))\big(D_t^2 + \tau^2\big)^{-N/2}u|_0 \\ &\qquad \leq |\big(D_t^2 + \tau^2\big)^{-N/2}P(x, D + i\tau\nabla\varphi(x))u|_0 + c(\lambda)|u|_{-N,m-1,\tau}\end{aligned}$$

which follows from the commutator rules for pseudodifferential operators. Furthermore, we have

$$\sqrt{\lambda}\,|D_t^m\big(D_t^2 + \tau^2\big)^{-N/2}u|_{0,\kappa^*} \leq \frac{c(\lambda)}{\sqrt{\tau}}|u|_{m-N,0,\tau}$$

and the corollary is proved. ∎

2.3. *Conclusion*

Without loss of generality we assume that ψ is analytic with respect to time in a complex neighborhood of I, in $I_{5r} \times iB_{5r}(0)$ for some $r > 0$.

Now we need to make the assumption that the differential operator has time-independent coefficients. We expand

$$P(x, D + i\tau\nabla\varphi(x)) = P(x, D) + \sum_{|\beta| \leq m-1} b_\beta(x)\tau^{m-|\beta|}D^\beta$$

and observe that b_β does not depend on τ but it will depend on λ. (More precisely, b_β is bounded for $\tau \to \infty$.) Furthermore, the coefficient b_β is analytic in time for $t \in I_{5r} \times iB_{5r}(0)$. When we like to emphasize the analyticity of b we will write $b(z, x')$.

We need one more pseudo-differential operator which derives from $P(x, D + i\tau\nabla\varphi(x))$. We set

$$B_\beta(z, x') = \chi(\operatorname{Re} z)\eta(\operatorname{Im} z)b_\beta(z, x'),$$

where $\chi \in C_0^\infty(I_{5r})$ with $\chi \equiv 1$ on I_{4r} and $\eta \in C_0^\infty(B_{5r}(0))$ with $\eta \equiv 1$ on $B_{4r}(0)$.

For $0 \leq \delta \leq 1$ we obtain a symbol $B_\beta(t + i\delta\xi_0, x') \in S^0(\mathbf{R} \times \mathbf{R})$ with values in $C^\infty(\overline{\Omega})$. We define

$$P_{\varphi,\delta}(x, D) = P(x, D) + \sum_{|\beta| \leq m-1} B_\beta^w(t + i\delta D_t, x')\tau^{m-|\beta|}D^\beta \tag{2.11}$$

and note that this operator is well defined due to the properties of $B_\beta(x)$ listed above. For a motivation we refer to Tataru [T2, Sect. 4].

The next lemma discusses the properties of the operator $P_{\varphi,\delta}$ with respect to commutation with the Gaussian regularizer

$$e^{-(\delta/2)D_t^2}w = \sqrt{\frac{1}{2\delta}}\int e^{-(1/2\delta)(t-s)^2}w(s,x')\,ds$$

and the fact that the operator $P_{\varphi,\delta}$ is a perturbation of $P(x, D + i\tau\nabla\varphi(x))$ in some sense. Let $\delta = 1/\tau$.

Lemma 2.7. *Let $P(x, D)$ be a differential operator on $I \times \Omega$ with time-independent bounded coefficients and let $\psi(x)$ be a function which is analytic with respect to time on $I_{5r} \times iB_{5r}(0)$ with values in $C^\infty(\overline{\Omega})$.*

Let $\chi \in C_0^\infty(I_{5r})$ such that $\chi \equiv 1$ on I_{4r}. Then

$$\left|\left(\chi(t)P_{\varphi,1/\tau}(x,D)e^{-(1/2\tau)D_t^2} - e^{-(1/2\tau)D_t^2}P(x, D + i\tau\nabla\varphi(x))\right)w\right|_0 \leq e^{-(r^2\tau/2)}|w|_{-N,m,\tau}$$

for $w \in C_0^\infty(I \times \Omega)$ and $\tau \geq \tau_4(N, \lambda, r)$ where N is a positive integer.

Second, we have

$$\left|\left(P_{\varphi,1/\tau}(x,D) - P(x, D + i\tau\nabla\varphi(x))\right)v\right|_0 \leq C(\lambda)|D_t v|_{m-1,\tau}$$

for $v \in C_0^\infty(I_{3r} \times \Omega)$.

Proof. The first estimate follows from [T2, Corollary 4.4]. Here the analyticity condition is critical. The second estimate follows from [T2, Proposition 4.5].

Note that the second estimate is better than [T2, Lemma 5.1]. This is due to the fact that the coefficients of the operator $P(x, D)$ are time independent. ∎

The following corollary studies the commutation properties of the Gaussian regularizer and the operator $P(x, D + i\tau\nabla\varphi(x))$.

Corollary 2.8. *Let $P(x, D)$ be a differential operator on $I \times \Omega$ with time-independent bounded coefficients and let $\psi(x)$ be a function which is analytic with respect to time on $I_{5r} \times iB_{5r}(0)$ with values in $C^\infty(\overline{\Omega})$. Suppose that $\mu \in C_0^\infty(I_{3r})$ is a cut off function such that $\mu \equiv 1$ on I_{2r}. Then*

$$|P(x, D + i\tau\nabla\varphi(x))\mu e^{-(1/2\tau)D_t^2}w - e^{-(1/2\tau)D_t^2}P(x, D + i\tau\nabla\varphi(x))w|_0 \leq \frac{C(\lambda, r)}{\tau}|\mu e^{-(1/2\tau)D_t^2}w|_{m,\tau} + 2e^{-(r^2\tau/4)}|w|_{-N,m,\tau} \tag{2.12}$$

provided $\tau \geq \tau_5(N, \lambda, r)$.

Proof. Using the triangle inequality we reduce the proof of the claim to an application of Lemma 2.7 and to the estimates

$$\left|(1-\chi)P_{\varphi,1/\tau}(x,D)\mu e^{-(1/2\tau)D_t^2}w\right|_0 \leq \frac{C(\lambda,r)}{\tau}|\mu e^{-(1/2\tau)D_t^2}w|_{m,\tau}$$

and

$$\left|\chi P_{\varphi,1/\tau}(x,D)(1-\mu)e^{-(1/2\tau)D_t^2}w\right|_0 \leq e^{-(r^2\tau/4)}|w|_{-N,m,\tau}$$

which follow from Corollary 3.5 and Corollary 3.3 which are both proved in the following section. ∎

Now we can finish the proof of Theorem 1.1. We substitute into (2.7)

$$v = \mu e^{-(1/2\tau)D_t^2}w,$$

where $w \in C_0^\infty(I \times \Omega)$ and μ is a cutoff function supported in I_{3r} which is identically 1 in I_{2r}. Applying the commutation estimate of the Gaussian regularizer and the operator $P(x, D + i\tau\nabla\varphi(x))$ given by formula (2.12) we have

$$\begin{aligned}
K\sqrt{\lambda}\,|\mu e^{-(1/2\tau)D_t^2}w|_{m,\kappa^*} &\leq |e^{-(1/2\tau)D_t^2}P(x,D+i\tau\nabla\varphi(x))w|_0 \\
&\quad + \sqrt{\lambda}\,|D_t^m\mu e^{-(1/2\tau)D_t^2}w|_{0,\kappa^*} + \frac{C(\lambda,r)}{\tau}|\mu e^{-(1/2\tau)D_t^2}w|_{m,\tau} \\
&\quad + 2e^{-(r^2\tau/4)}|w|_{-N,m,\tau}.
\end{aligned}$$

At first we will transfer the third term on the right hand side into the left hand side. This can be done since

$$\begin{aligned}
&\lambda K^2\kappa(x)^{2m-2|\alpha|-1} - 2C(\lambda,r)^2\tau^{2m-2|\alpha|-2} \\
&\quad = \lambda K^2\kappa(x)^{2m-2|\alpha|-1}\left(1 - 2\frac{C(\lambda,r)^2}{\tau K^2\lambda^{2m-2|\alpha|}e^{(2m-2|\alpha|-1)\lambda\psi(x)}}\right) \\
&\quad \geq \lambda\frac{K^2}{2}\kappa(x)^{2m-2|\alpha|-1}
\end{aligned}$$

provided

$$\tau \geq \tau_6(\lambda) = \max_{|\alpha| \leq m} 4 \frac{C(\lambda, r)^2}{K^2 \lambda^{2m-2|\alpha|} e^{(2m-2|\alpha|-1)\lambda\psi_{min}}}.$$

Thus,

$$\begin{aligned}
&\frac{1}{2} K\sqrt{\lambda}\, | \mu e^{-(1/2\tau)D_t^2} w|_{m,\kappa^*} \\
&\qquad \leq |e^{-(1/2\tau)D_t^2} P(x, D + i\tau\nabla\varphi(x)) w|_0 \\
&\qquad\qquad + \sqrt{\lambda}\, |D_t^m \mu e^{-(1/2\tau)D_t^2} w|_{0,\kappa^*} + 2e^{-(r^2\tau/4)} |w|_{-N,m,\tau}. \qquad (2.13)
\end{aligned}$$

Next we use

$$e^{-(1/2\tau)D_t^2} w = \mu e^{-(1/2\tau)D_t^2} w + (1-\mu) e^{-(1/2\tau)D_t^2} w$$

and Corollary 3.2 gives

$$|e^{-(1/2\tau)D_t^2} w|_{m,\kappa^*} \leq | \mu e^{-(1/2\tau)D_t^2} w|_{m,\kappa^*} + e^{-(\tau r^2/4)} |w|^2_{-N,m,\tau}$$

for $\tau \geq \tau_7(N, r, \lambda)$.

We need to get a similar estimate for the second term in the right hand side of (2.13). Relying on a similar argument we have

$$\begin{aligned}
\sqrt{\lambda}\, |D_t^m \mu e^{-(1/2\tau)D_t^2} w|_{0,\kappa^*} &\leq \sqrt{\frac{\lambda}{\kappa_{min}}}\, |D_t^m \mu e^{-(1/2\tau)D_t^2} w|_0 \\
&\leq \sqrt{\frac{\lambda}{\kappa_{min}}}\, |D_t^m e^{-(1/2\tau)D_t^2} w|_0 + e^{-(r^2\tau/4)} |w|_{-N,0,\tau},
\end{aligned}$$

where κ_{min} denotes the minimum value of κ on $I_{3r} \times \Omega$. Hence, combining the previous two estimates with (2.13) we obtain

$$\begin{aligned}
\frac{1}{2} K\sqrt{\lambda}\, |e^{-(1/2\tau)D_t^2} w|_{m,\kappa^*} &\leq |e^{-(1/2\tau)D_t^2} P(x, D + i\tau\nabla\varphi(x)) w|_0 \\
&\qquad + \sqrt{\frac{\lambda}{\kappa_{min}}}\, |D_t^m e^{-(1/2\tau)D_t^2} w|_0 \\
&\qquad + \left(3 + \sqrt{\lambda} K/2\right) e^{-(r^2\tau/4)} |w|_{-N,m,\tau}.
\end{aligned}$$

We estimate the second term in the right hand side

$$
\begin{aligned}
|D_t^m e^{-(1/2\tau)D_t^2} w|_0^2 &= \int \xi_0^{2m} e^{-(1/\tau)\xi_0^2} |\hat{w}(\xi)|^2 \, d\xi \\
&= \int_{\xi_0^2 \le \varepsilon\tau^2} \xi_0^{2m} e^{-(1/\tau)\xi_0^2} |\hat{w}(\xi)|^2 \, d\xi \\
&\quad + \int_{\xi_0^2 > \varepsilon\tau^2} \xi_0^{2m} e^{-(1/\tau)\xi_0^2} |\hat{w}(\xi)|^2 \, d\xi \\
&\le \varepsilon^m \tau^{2m} \int_{\xi_0^2 \le \varepsilon\tau^2} e^{-(1/\tau)\xi_0^2} |\hat{w}(\xi)|^2 \, d\xi \\
&\quad + \sup_{\xi_0^2 > \varepsilon\tau^2} e^{-(1/\tau)\xi_0^2} \left(\xi_0^2 + \tau^2 \right)^{N+m} \\
&\qquad \times \int \left(\xi_0^2 + \tau^2 \right)^{-N-m} \xi_0^{2m} |\hat{w}(\xi)|^2 \, d\xi \\
&\le \varepsilon^m \tau^{2m} |e^{-(1/2\tau)D_t^2} w|_0^2 + e^{-(\varepsilon\tau/2)} |w|_{-N,0,\tau}^2
\end{aligned}
$$

for $\tau \ge \tau_8(\varepsilon, N)$. Hence, we have

$$
\begin{aligned}
&\frac{1}{2} K\sqrt{\lambda}\, |e^{-(1/2\tau)D_t^2} w|_{m,\kappa^*} \\
&\quad \le |e^{-(1/2\tau)D_t^2} P(x, D + i\tau\nabla\varphi(x)) w|_0 \\
&\qquad + \sqrt{\frac{\lambda}{\kappa_{min}}}\, \varepsilon^{m/2} \tau^m |e^{-(1/2\tau)D_t^2} w|_0 \\
&\qquad + \left(3 + \sqrt{\lambda}\, K/2\right) e^{-(r^2\tau/4)} |w|_{-N,m,\tau} + e^{-(\varepsilon\tau/4)} |w|_{-N,0,\tau}.
\end{aligned}
$$

The second term on the right hand side can be controlled by the left hand side provided $\varepsilon = \varepsilon(\lambda)$ is small. This follows from

$$
\begin{aligned}
&\frac{1}{4} K^2 \lambda \kappa(x)^{2m-1} - 2\frac{\lambda}{\kappa_{min}} \varepsilon^m \tau^{2m} \\
&\quad = \frac{1}{4} K^2 \lambda \kappa(x)^{2m-1} \left(1 - 8 \frac{\varepsilon^m}{e^{\lambda\psi_{min}} K^2 \lambda^{2m} e^{(2m-1)\lambda\psi(x)}} \right) \\
&\quad \ge \frac{1}{8} K^2 \lambda \kappa(x)^{2m-1}
\end{aligned}
$$

if

$$\varepsilon = \varepsilon(\lambda) = \frac{1}{16}\lambda^2 e^{2\lambda\psi_{min}} K^{2/m}.$$

Thus, we arrive at

$$\frac{1}{4}K\sqrt{\lambda}\,|e^{-(1/2\tau)D_t^2}w|_{m,\kappa^*} \leq |e^{-(1/2\tau)D_t^2}P(x, D + i\tau\nabla\varphi(x))w|_0$$
$$+ \left(3 + \sqrt{\lambda}\,K/2\right)e^{-(r^2\tau/4)}|w|_{-N,m,\tau}$$
$$+ e^{-(\varepsilon(\lambda)\tau/4)}|w|_{-N,0,\tau}.$$

Finally, we observe that Corollary 2.6 implies

$$|w|_{-N,m,\tau} \leq c(\lambda)\left(\sqrt{\tau}|P(x, D + i\tau\nabla\varphi(x))w|_0 + |w|_{-N+m,m-1,\tau}\right)$$

and choosing $N = m$ and

$$d(\lambda) = \min\left\{\frac{r^2}{8}, \frac{\varepsilon(\lambda)}{8}\right\}$$

yields

$$\frac{1}{\sqrt{8}}K\sqrt{\lambda}\,|e^{-(1/2\tau)D_t^2}w|_{m,\kappa^*} \leq |e^{-(1/2\tau)D_t^2}P(x, D + i\tau\nabla\varphi(x))w|_0$$
$$+ e^{-d(\lambda)\tau}|P(x, D + i\tau\nabla\varphi(x))w|_0$$
$$+ e^{-d(\lambda)\tau}|w|_{m-1,\tau}$$

for $\tau \geq \tau_0(\lambda) = \max_{j=3\dots8}\{\tau_j(\lambda)\}$. Theorem 1.1 is proved.

For the proof of Corollary 1.2 we have to do the same procedure. However, our starting point will be Corollary 2.5.

3. SOME TECHNICAL LEMMAS

Here we state and prove the technical results needed in the proof of Theorem 1.1. The first results are concerned with the exponential decay of the Gaussian regularizer off the diagonal.

LEMMA 3.1. *Let $\mu(t) \in C_0^\infty(I_{3r})$ be a cut off function such that $\mu \equiv 1$ on I_{2r} and let w be a distribution with compact support in I. Then*

$$|(1 - \mu)e^{-(1/\tau)D_t^2}w|_0 \leq e^{-(r^2\tau/4)}|w|_{-N,\tau}$$

for $\tau \geq \tau(N, r)$ provided the right hand side is finite.

Proof. Since w is compactly supported in I we have

$$(1-\mu(t))e^{-(1/2\tau)D_t^2}w = \int (1-\mu(t))\,\nu(s)e^{-(\tau/2)(t-s)^2}w(s)\,ds,$$

where $\nu \in C_0^\infty(I_r)$ such that $\nu \equiv 1$ on I. (Of course, the integral has to be understood in the sense of distributions.) Then we have

$$\begin{aligned}|(1-\mu)e^{-(1/2\tau)D_t^2}w|_0^2 &= \int \left| \int (1-\mu(t))\,\nu(s)e^{-(\tau/2)(t-s)^2}w(s)\,ds \right|^2 dt \\ &\le \int_{\mathbf{R}\setminus I_{2r}} \left| \int \nu(s)e^{-(\tau/2)(t-s)^2}w(s)\,ds \right|^2 dt \\ &\le \int_{\mathbf{R}\setminus I_{2r}} |\nu e^{-(\tau/2)(t-\cdot)^2}|_{N,\tau}^2\,dt\,|w|_{-N,\tau}^2,\end{aligned}$$

where we used $\operatorname{supp}(1-\mu) \subset \mathbf{R}\setminus I_{2r}$ and duality in Sobolev spaces. Next, we estimate the norm under the integral sign using [H3, Lemma 3.2]. We obtain

$$\begin{aligned}|\nu e^{-(\tau/2)(t-\cdot)^2}|_{N,\tau} &\le C(N)|\nu|_{N,\tau} \max_{k\le N} \tau^{-k} \sup_{s\in I_r} |\partial_s^k e^{-(\tau/2)(t-s)^2}| \\ &\le C(N,r)\tau^N \max_{k\le N} \tau^{-k} \sup_{s\in I_r} |\partial_s^k e^{-(\tau/2)(t-s)^2}| \\ &\le C(N,r)\tau^N \sup_{s\in I_r} (1+|t-s|)^N e^{-(\tau/2)(t-s)^2}\end{aligned}$$

which leads to

$$\begin{aligned}|(1-\mu)e^{-(1/2\tau)D_t^2}w|_0^2 &\le C(N,r)\tau^{2N}(1+r)^{2N}e^{-r^2\tau}|w|_{-N,\tau}^2 \\ &\le e^{-(r^2\tau/2)}|w|_{-N,\tau}^2\end{aligned}$$

provided $\tau \ge \tau(N,r)$. ∎

In the same manner we can prove the following estimate.

COROLLARY 3.2. *Let μ be given as in Lemma 3.1 and let w be a distribution compactly supported in I with values in $H^m(\Omega)$. Then*

$$|(1-\mu)e^{-(1/2\tau)D_t^2}w|_{m,\kappa^*} \le e^{-(r^2\tau/4)}|w|_{-N,m,\tau}$$

for $\tau \ge \tau(N,r,\lambda)$ provided the right hand side is finite.

Using the fact that operator $P_{\varphi,1/\tau}(x,D)$ is a continuous operator from $H^m(\mathbf{R}\times\Omega)$ into $L_2(\mathbf{R}\times\Omega)$ (see [T2, Proposition 4.5]) uniformly in τ for $\tau\geq 1$ we can show the following result.

COROLLARY 3.3. *Let μ be given as in Lemma 3.1 and let w be a distribution compactly supported in I with values in $H^m(\Omega)$. Moreover, let $\chi(t)\in C_0^\infty(I_{5r})$ be a cut off function with $\chi\equiv 1$ on I_{4r}. Then*

$$|\chi P_{\varphi,1/\tau}(x,D)(1-\mu)e^{-(1/2\tau)D_t^2}w|_0 \leq e^{-(r^2\tau/4)}|w|_{-N,m,\tau}$$

for $\tau\geq\tau(N,r,\lambda)$ provided the right hand side is finite.

Finally, we need a result which describes the decay of the kernel of the operator $P_{\varphi,1/\tau}(x,D)$ off the diagonal.

LEMMA 3.4. *Let $\chi(t)\in C_0^\infty(I_{5r})$ be a cut off function with $\chi\equiv 1$ on I_{4r}. Furthermore, let $a(t,\xi)\in S^0(\mathbf{R}\times\mathbf{R})$. Then, for every positive integer $M\geq 2$*

$$|(1-\chi)a^w(x,\delta D)v|_0 \leq \frac{2C(M)}{(M-1)^2}\left(\frac{\delta}{r}\right)^{M-1}|v|_0$$

for $v\in C_0^\infty(I_{3r})$ and $0\leq\delta\leq 1$.

Proof. The kernel of the operator $(1-\chi(t))a^w(t,\delta D_t)$ is

$$K(s,t) = \frac{1}{2\pi}(1-\chi(t))\int a\left(\frac{t+s}{2},\delta\xi\right)e^{i(t-s)\xi}\,d\xi.$$

This is an oscillatory integral which can be regularized since it is zero on the diagonal. For any positive integer $M\geq 2$ we have

$$K(s,t) = \frac{1}{2\pi}(1-\chi(t))\int D_\xi^M a\left(\frac{t+s}{2},\delta\xi\right)\frac{1}{(s-t)^M}e^{i(t-s)\xi}\,d\xi. \quad (3.1)$$

Since

$$\left|D_\xi^M a\left(\frac{t+s}{2},\delta\xi\right)\right| \leq C(M)(1+|\delta\xi|)^{-M}\delta^M$$

by assumption, we obtain

$$\int\left|D_\xi^M a\left(\frac{t+s}{2},\delta\xi\right)\right|d\xi \leq \frac{C(M)\delta^{M-1}}{M-1}.$$

We will prove the claim of the lemma using Schur's theorem [H2, Lemma 18.1.12]. For that purpose we have to estimate

$$\sup_{s\in I_{3r}}\int|K(s,t)|\,dt \qquad\text{and}\qquad \sup_{t\in\mathbf{R}}\int|K(s,t)|\,ds.$$

Using the representation of the kernel given in (3.1) and the estimate on the integral with respect to ξ we obtain

$$\sup_{s\in I_{3r}} \int |K(s,t)|\, dt \le \frac{C(M)\,\delta^{M-1}}{M-1} \sup_{s\in I_{3r}} \int_{\mathbf{R}\setminus I_{4r}} \frac{dt}{|s-t|^M}$$
$$\le \frac{2C(M)}{(M-1)^2}\left(\frac{\delta}{r}\right)^{M-1}$$

and

$$\sup_{t\in \mathbf{R}} \int |K(s,t)|\, ds \le \frac{C(M)\,\delta^{M-1}}{M-1} \sup_{t\in \mathbf{R}\setminus I_{4r}} \int_{I_{3r}} \frac{ds}{|s-t|^M}$$
$$\le \frac{2C(M)}{(M-1)^2}\left(\frac{\delta}{r}\right)^{M-1}.$$

∎

Setting $M = 2$ and using the continuity properties of $P_{\varphi, 1/\tau}(x, D)$ we have

COROLLARY 3.5. *Let χ and μ be defined as in Lemma* 3.4. *Then*

$$|(1-\chi)P_{\varphi,1/\tau}(x,D)v|_0 \le \frac{C(\lambda, r)}{\tau}|v|_{m,\tau}$$

for $v \in C_0^\infty(I_{3r} \times \Omega)$.

4. FURTHER RESULTS

Here we state some Carleman estimates which can be derived in a similar manner as Proposition 2.4. In contrast to the previous sections we will not distinguish the time variable and assume Ω is a bounded domain in $\mathbf{R}^n$. The following theorems will clarify under which condition the Carleman estimate carries an additional large parameter. More specifically, if the level surfaces of ψ satisfy the assumptions of Calderon's uniqueness theorem [H2, Theorem 28.1.1] we obtain an estimate with an additional parameter.

THEOREM 4.1. *Let $P(x, D)$ be a differential operator on Ω of order m with C^∞ coefficients in the principal part and bounded coefficients otherwise. Moreover, let $\psi(x) \in C^\infty(\overline{\Omega})$ with $\nabla\psi(x) \neq 0$ such that $p_\psi(x, \xi, \eta)$ considered as a polynomial in η has no multiple (complex) root with $\xi + i\eta\nabla\psi(x) \neq 0$ for $\xi \in \mathbf{R}^n$ and $x \in \overline{\Omega}$.*

If $p(x, \xi)$ has real coefficients then there exists a constant $c > 0$ such that

$$\sqrt{\lambda}\,|\kappa^{m-|\alpha|-1/2} D^{\alpha}(e^{\tau\varphi}u)|_0 \le c|e^{\tau\varphi}P(x, D)u|_0, \qquad |\alpha| \le m - 1$$

for $u \in H^{m-1}$ compactly supported in Ω provided $\lambda \ge \lambda_0$ and $\tau \ge \tau_0(\lambda)$ and the right hand side is finite. Here $\kappa(x)$ is the function already introduced in Section 1, i.e., $\kappa = \tau\lambda e^{\lambda\psi}$.

Moreover, if $p(x, \xi)$ is elliptic then there exists a constant $c > 0$ such that

$$\sqrt{\lambda}\,|\kappa^{m-|\alpha|-1/2} D^{\alpha}(e^{\tau\varphi}u)|_0 \le c|e^{\tau\varphi}P(x, D)u|_0, \qquad |\alpha| \le m$$

for $u \in H^{m-1}$ compactly supported in Ω provided $\lambda \ge \lambda_0$ and $\tau \ge \tau_0(\lambda)$ and the right hand side is finite.

On the other hand, if the level surfaces of ψ are only strongly pseudo-convex with respect to the operator [H2, Definition 28.3.1] we do not obtain the additional parameter up front.

THEOREM 4.2. *Let $P(x, D)$ be a differential operator on Ω of order m with C^{∞} coefficients in the principal part and bounded coefficients otherwise. Moreover, assume that $\psi(x) \in C^{\infty}(\overline{\Omega})$ has strongly pseudo-convex level surfaces with respect to $P(x, D)$ in $\overline{\Omega}$.*

If $p(x, \xi)$ has real coefficients then there exists a constant $c > 0$ such that

$$|\kappa^{m-|\alpha|-1/2} D^{\alpha}(e^{\tau\varphi}u)|_0 \le c|e^{\tau\varphi}P(x, D)u|_0, \qquad |\alpha| \le m - 1$$

for $u \in H^{m-1}$ compactly supported in Ω provided $\lambda \ge \lambda_0$ and $\tau \ge \tau_0(\lambda)$ and the right hand side is finite.

Moreover, if $p(x, \xi)$ is elliptic then there exists a constant $c > 0$ such that

$$|\kappa^{m-|\alpha|-1/2} D^{\alpha}(e^{\tau\varphi}u)|_0 \le c|e^{\tau\varphi}P(x, D)u|_0, \qquad |\alpha| \le m$$

for $u \in H^{m-1}$ compactly supported in Ω provided $\lambda \ge \lambda_0$ and $\tau \ge \tau_0(\lambda)$ and the right hand side is finite.

These two results clarify the influence of the second parameter λ in the Carleman estimate. In a certain sense they generalize the results given by Isakov [I].

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